

2-Distance Colorings of Integer Distance Graphs

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Abstract

A 2-distance k -coloring of a graph G is a mapping from $V(G)$ to the set of colors $\{1, \dots, k\}$ such that every two vertices at distance at most 2 receive distinct colors. The 2-distance chromatic number $\chi_2(G)$ of G is then the smallest k for which G admits a 2-distance k -coloring. For any finite set of positive integers $D = \{d_1, \dots, d_k\}$, the integer distance graph $G = G(D)$ is the infinite graph defined by $V(G) = \mathbb{Z}$ and $uv \in E(G)$ if and only if $|v - u| \in D$. We study the 2-distance chromatic number of integer distance graphs for several types of sets D . In each case, we provide exact values or upper bounds on this parameter and characterize those graphs $G(D)$ with $\chi_2(G(D)) = \Delta(G(D)) + 1$.

Keywords: 2-distance coloring; Integer distance graph.

MSC 2010: 05C15, 05C12.

1 Introduction

All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G , respectively. For any two vertices u and v of G , we denote by $d_G(u, v)$ the *distance* between u and v , that is the length of a shortest path joining u and v . We denote by $\Delta(G)$ the maximum degree of G .

A (proper) k -coloring of a graph G is a mapping from $V(G)$ to the set of colors $\{1, \dots, k\}$ such that every two adjacent vertices receive distinct colors. The smallest k for which G admits a k -coloring is the *chromatic number* of G , denoted $\chi(G)$. A 2-distance k -coloring of a graph G is a mapping from $V(G)$ to the set of colors $\{1, \dots, k\}$ such that every two vertices at distance at most 2 receive distinct colors. 2-distance colorings are sometimes called $L(1,1)$ -labelings (see [5] for a survey on $L(h,k)$ -labelings) or *square colorings* in the literature. The smallest k for which G admits a 2-distance k -coloring is the *2-distance chromatic number* of G , denoted $\chi_2(G)$.

The *square* G^2 of a graph G is the graph defined by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $d_G(u, v) \leq 2$. Clearly, a 2-distance coloring of a graph G is nothing but a proper coloring of G^2 and, therefore, $\chi_2(G) = \chi(G^2)$ for every graph G .

The study of 2-distance colorings was initiated by Kramer and Kramer [7] (see also their survey on general distance colorings in [8]). The case of planar graphs has attracted a lot of attention in the literature (see e.g. [1, 2, 3, 4, 6, 9, 12]), due to the conjecture of Wegner that suggests an upper bound on the 2-distance chromatic number of planar graphs depending on their maximum degree (see [13] for more details).

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In this paper, we study 2-distance colorings of distance graphs. Although several coloring problems have been considered for distance graphs (see [10] for a survey), it seems that 2-distance colorings have not been considered yet. We present in Section 2 a few basic results on the chromatic number of distance graphs. We then consider specific sets D , namely $D = \{1, a\}$, $a \geq 3$ (in Section 3), $D = \{1, a, a+1\}$, $a \geq 3$ (in Section 4), and $D = \{1, \dots, m, a\}$, $2 \leq m < a$ (in Section 5). We finally propose some open problems in Section 6.

2 Preliminaries

Let $D = \{d_1, \dots, d_k\}$ be a finite set of positive integers. The *integer distance graph* (simply called *distance graph* in the following) $G = G(D)$ is the infinite graph defined by $V(G) = \mathbb{Z}$ and $uv \in E(G)$ if and only if $|v - u| \in D$.

If $\gcd(\{d_1, \dots, d_k\}) = p > 1$, the distance graph $G(D)$ has p connected components, each of them being isomorphic to the distance graph $G(D')$ with $D' = \{d_1/p, \dots, d_k/p\}$. In that case, we thus have $\chi_2(G(D)) = \chi_2(G(D'))$ so that we can always assume $\gcd(D) = 1$.

It is easy to observe that the square of the distance graph $G(D)$ is also a distance graph, namely the distance graph $G(D^2)$ where

$$D^2 = D \cup \{d + d' \mid d, d' \in D\} \cup \{d - d' \mid d, d' \in D, d > d'\}.$$

For instance, for $D = \{1, 2, 5\}$, we get $D^2 = \{1, 2, 3, 4, 5, 6, 7, 10\}$. Note that if D has cardinality k , then D^2 has cardinality at most $k(k+1)$.

As observed in the previous section, $\chi_2(G) = \chi(G^2)$ for every graph G . Therefore, since $(G(D))^2 = G(D^2)$, determining the 2-distance chromatic number of the distance graph $G(D)$ reduces to determining the chromatic number of the distance graph $G(D^2)$. The problem of determining the chromatic number of distance graphs has been extensively studied in the literature. When $|D| \leq 2$, this question is easily solved, thanks to the following general upper bounds:

Proposition 1 (folklore) *For every finite set of positive integers $D = \{d_1, \dots, d_k\}$ and every positive integer p such that $d_i \not\equiv 0 \pmod{p}$ for every i , $1 \leq i \leq k$, $\chi(G(D)) \leq p$.*

Proof. Let $\lambda : V(G(D)) \rightarrow \{1, \dots, p\}$ be the mapping defined by

$$\lambda(x) = 1 + (x \pmod{p}),$$

for every integer $x \in \mathbb{Z}$. Since $d_i \not\equiv 0 \pmod{p}$ for every i , $1 \leq i \leq k$, the mapping λ is clearly a proper coloring of $G(D)$. \square

Theorem 2 (Walther [11]) *For every finite set of positive integers D ,*

$$\chi(G(D)) \leq |D| + 1.$$

Proof. A $(|D|+1)$ -coloring of $G(D)$ can easily be produced using the First-Fit greedy algorithm, starting from vertex 0, from left to right and then from right to left, since every vertex has exactly $|D|$ neighbors on its left and $|D|$ neighbors on its right. \square

Therefore, when $|D| \leq 2$, $\chi(G(D)) = 2$ if $|D| = 1$ or all elements in D are odd (since $G(D)$ is then bipartite), and $\chi(G(D)) = 3$ otherwise (since $G(D)$ then contains cycles of odd length). The case $|D| = 3$ has been settled by Zhu [14]. Whenever $|D| \geq 4$, only partial results have been obtained, namely for sets D having specific properties.

A coloring λ of a distance graph $G(D)$ is *p-periodic*, for some integer $p \geq 1$, if $\lambda(x+p) = \lambda(x)$ for every $x \in \mathbb{Z}$. Walther also proved the following:

Theorem 3 (Walther [11]) *For every finite set of positive integers D , if $\chi(G(D)) \leq k$ then $G(D)$ admits a p -periodic k -coloring for some p .*

The *pattern* of such a p -periodic coloring is defined as the sequence $\lambda(x) \dots \lambda(x+p-1)$. In particular, the coloring defined in the proof of Proposition 1 was p -periodic with pattern $12 \dots p$. In the following, we will describe such patterns using standard notation of Combinatorics on words. For instance, the pattern 121212345 will be denoted $(12)^3 345$.

Finally, note that in any 2-distance coloring of a graph G , all vertices in the closed neighborhood of any vertex must be assigned distinct colors. Therefore, we have the following:

Observation 4 *For every graph G , $\chi_2(G) \geq \Delta(G) + 1$.*

In particular, this bound is attained by the distance graph $G(D)$ with $D = \{1, \dots, k\}$, $k \geq 2$:

Proposition 5 *For every $k \geq 2$, $\chi_2(G(\{1, \dots, k\})) = 2k + 1 = \Delta(G(\{1, \dots, k\})) + 1$.*

Proof. It is easy to check that the mapping λ given by

$$\lambda(x) = 1 + (x \bmod 2k + 1)$$

for every $x \in \mathbb{Z}$ is a 2-distance $(2k + 1)$ -coloring of $G(\{1, \dots, k\})$. Equality then follows from Observation 4. \square

3 The case $D = \{1, a\}$, $a \geq 3$

We study in this section the 2-distance chromatic number of distance graphs $G(D)$ with $D = \{1, a\}$, $a \geq 3$ (note that the case $a = 2$ is already solved by Proposition 5).

When $D = \{1, a\}$, $a \geq 3$, we have $\Delta(G(D)) = 4$ and

$$D^2 = \{1, 2, a - 1, a, a + 1, 2a\}.$$

The following theorem gives the 2-distance chromatic number of any such graph:

Theorem 6 *For every integer $a \geq 3$,*

$$\chi_2(G(\{1, a\})) = \begin{cases} 5 & \text{if } a \equiv 2 \pmod{5}, \text{ or } a \equiv 3 \pmod{5}, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. Since $\{1, a\}^2 = \{1, 2, a - 1, a, a + 1, 2a\}$, we get $d \not\equiv 0 \pmod{5}$ for every $d \in \{1, a\}^2$ whenever $a \equiv 2 \pmod{5}$ or $a \equiv 3 \pmod{5}$ and thus, by Proposition 1 and Observation 4, $\chi_2(G(\{1, a\})) = 5$.

Note that for every $x \in \mathbb{Z}$, the set of vertices

$$C(x) = \{x - a, x - 1, x, x + 1, x + a\}$$

induces a clique in $G(\{1, a\}^2)$ (see Figure 1). We now claim that every 2-distance 5-coloring λ of $G(\{1, a\})$ is necessarily 5-periodic, that is $\lambda(x + 5) = \lambda(x)$ for every $x \in \mathbb{Z}$. To show that, it suffices to prove that any five consecutive vertices $x, \dots, x + 4$ must be assigned distinct colors. Assume to the contrary that this is not the case and, without loss of generality, let $x = 0$. Since vertices 0, 1 and 2 necessarily get distinct colors, we only have two cases to consider:

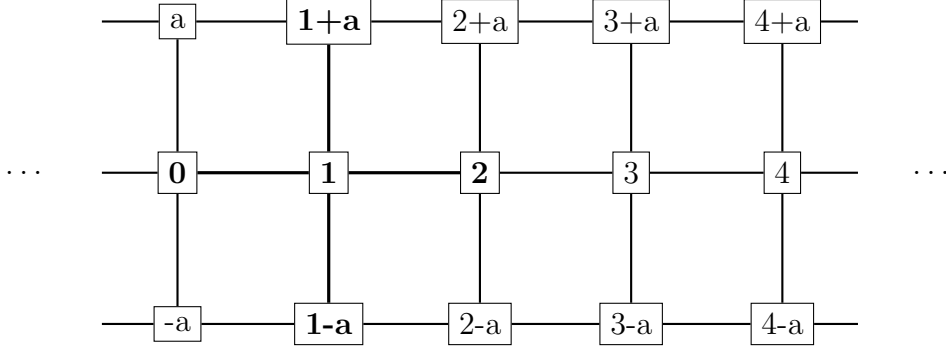


Figure 1: The distance graph $G(\{1, a\})$, $a \geq 3$

1. $\lambda(0) = \lambda(3) = 1$, $\lambda(1) = 2$, $\lambda(2) = 3$.

Since $C(1)$ induces a clique in $G(\{1, a\}^2)$ (depicted in bold in Figure 1), we have

$$\{\lambda(1-a), \lambda(1+a)\} = \{4, 5\},$$

which implies

$$\{\lambda(2-a), \lambda(2+a)\} = \{4, 5\}.$$

(More precisely, $\lambda(2-a) = 9 - \lambda(1-a)$ and $\lambda(2+a) = 9 - \lambda(1+a)$). This implies $\lambda(3-a) = \lambda(3+a) = 2$, a contradiction since $d(3-a, 3+a) = 2$.

2. $\lambda(0) = \lambda(4) = 1$, $\lambda(1) = 2$, $\lambda(2) = 3$, $\lambda(3) = 4$.

As in the previous case we have

$$\{\lambda(1-a), \lambda(1+a)\} = \{4, 5\},$$

which implies

$$\{\lambda(2-a), \lambda(2+a)\} = \{1, 5\}.$$

We then get $\lambda(3-a) = \lambda(3+a) = 2$, again a contradiction.

Therefore, $\chi_2(G(\{1, a\})) = 5$ if and only if 5 do not divide any element of $\{1, a\}^2 = \{1, 2, a-1, a, a+1, 2a\}$. This is clearly the case if and only if $a \equiv 2 \pmod{5}$ or $a \equiv 3 \pmod{5}$.

We finally prove that there exists a 2-distance 6-coloring of $G(\{1, a\})$ for any value of a . We consider three cases, according to the value of $(a \pmod{3})$:

1. $a = 3k$, $k \geq 1$.

Let λ be the $(2a-1)$ -periodic mapping defined by the pattern

$$(123)^k (456)^{k-1} 45.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 5$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k-1\} \cup \{(2a-1)p+3q, p \geq 1, 1-k \leq q \leq k-1\}.$$

If $\lambda(x) = \lambda(y) = 6$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k-2\} \cup \{(2a-1)p+3q, p \geq 1, 2-k \leq q \leq k-2\}.$$

Therefore, in both cases, $d(x, y) \notin \{1, 2, a-1, a, a+1, 2a\}$, and thus λ is a 2-distance 6-coloring of $G(\{1, a\})$.

2. $a = 3k + 1, k \geq 1$.

Let λ be the $(2a - 2)$ -periodic mapping defined by the pattern

$$(123)^k(456)^k.$$

If $\lambda(x) = \lambda(y) = c, 1 \leq c \leq 6$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k - 1\} \cup \{(2a - 2)p + 3q, p \geq 1, 1 - k \leq q \leq k - 1\}.$$

Therefore, $d(x, y) \notin \{1, 2, a - 1, a, a + 1, 2a\}$, and thus λ is a 2-distance 6-coloring of $G(\{1, a\})$.

3. $a = 3k + 2, k \geq 1$.

Let λ be the $(2a + 1)$ -periodic mapping defined by the pattern

$$(123)^{k+1}(456)^k 45.$$

If $\lambda(x) = \lambda(y) = c, 1 \leq c \leq 5$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k\} \cup \{(2a + 1)p + 3q, p \geq 1, -k \leq q \leq k\}.$$

If $\lambda(x) = \lambda(y) = 6$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k - 1\} \cup \{(2a + 1)p + 3q, p \geq 1, 1 - k \leq q \leq k - 1\}.$$

Therefore, in both cases, $d(x, y) \notin \{1, 2, a - 1, a, a + 1, 2a\}$, and thus λ is a 2-distance 6-coloring of $G(\{1, a\})$.

This concludes the proof. □

4 The case $D = \{1, a, a + 1\}, a \geq 3$

We study in this section the 2-distance chromatic number of distance graphs $G(D)$ with $D = \{1, a, a + 1\}, a \geq 3$ (note that the case $a = 2$ is already solved by Proposition 5).

When $D = \{1, a, a + 1\}, a \geq 3$, we have $\Delta(G(D)) = 6$ and

$$D^2 = \{1, 2, a - 1, a, a + 1, a + 2, 2a, 2a + 1, 2a + 2\}.$$

We first prove the following:

Theorem 7 *For every integer $a, a \geq 3$,*

$$\chi_2(G(\{1, a, a + 1\})) = 7 = \Delta(G(\{1, a, a + 1\})) + 1$$

if and only if $a \equiv 2 \pmod{7}$ or $a \equiv 4 \pmod{7}$.

Proof. Since $\{1, a, a + 1\}^2 = \{1, 2, a - 1, a, a + 1, a + 2, 2a, 2a + 1, 2a + 2\}$, we get $d \not\equiv 0 \pmod{7}$ for every $d \in \{1, a, a + 1\}^2$ whenever $a \equiv 2 \pmod{7}$ or $a \equiv 4 \pmod{7}$ and thus, by Proposition 1 and Observation 4, $\chi_2(G(\{1, a, a + 1\})) = 7$.

Note that for every $x \in \mathbb{Z}$, the set of vertices

$$C(x) = \{x - a - 1, x - a, x - 1, x, x + 1, x + a, x + a + 1\}$$

induces a clique in $G(\{1, a, a + 1\}^2)$. We now claim that every 2-distance 7-coloring λ of $G(\{1, a, a + 1\})$ is necessarily 7-periodic, that is $\lambda(x + 7) = \lambda(x)$ for every $x \in \mathbb{Z}$. To show that, it suffices to prove that any 7 consecutive vertices $x, \dots, x + 6$ must be assigned distinct colors. Assume to the contrary that this is not the case and, without loss of generality, let $x = 0$. Since vertices 0, 1 and 2 necessarily get distinct colors, we only have four cases to consider (see Figure 2):

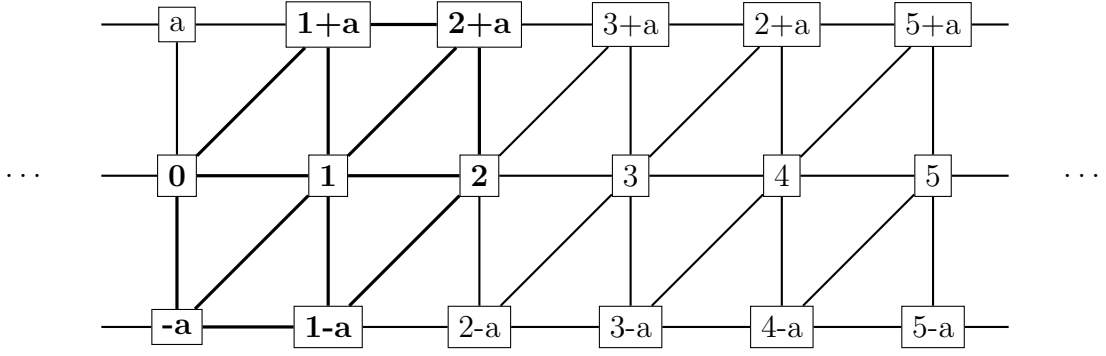


Figure 2: The distance graph $G(\{1, a, a+1\})$, $a \geq 3$

1. Vertices 0, 1, 2, 3 are colored with the colors 1, 2, 3 and 1, respectively.

We consider two subcases:

- (a) $\lambda(4) = 2$.

Since $C(1)$ induces a clique in $G(\{1, a, a+1\}^2)$ (depicted in bold in Figure 2), we have

$$\{\lambda(-a), \lambda(1-a), \lambda(1+a), \lambda(2+a)\} = \{4, 5, 6, 7\}.$$

For similar reasons, we also have

$$\{\lambda(2-a), \lambda(3-a), \lambda(3+a), \lambda(4+a)\} = \{4, 5, 6, 7\}.$$

This implies $\lambda(-a) = \lambda(4-a)$ or $\lambda(1+a) = \lambda(5+a)$. Each of these cases thus corresponds to case 2 below.

- (b) $\lambda(4) \notin \{1, 2, 3\}$.

Assume $\lambda(4) = 4$, without loss of generality. As in the previous subcase, we have

$$\{\lambda(-a), \lambda(1-a), \lambda(1+a), \lambda(2+a)\} = \{4, 5, 6, 7\},$$

and, similarly,

$$\{\lambda(1-a), \lambda(2-a), \lambda(2+a), \lambda(3+a)\} = \{4, 5, 6, 7\}.$$

Moreover, since $\lambda(4) = 4$, we get

$$\{\lambda(3+a), \lambda(2-a)\} \subseteq \{5, 6, 7\}.$$

On the other hand, considering the clique $S(3)$ in $G(\{1, a, a+1\}^2)$, we also get

$$\{\lambda(4+a), \lambda(3-a)\} \subseteq \{5, 6, 7\}.$$

We thus get a contradiction since we only have three available colors for the clique induced by the four vertices $2-a$, $3-a$, $a+3$ and $a+4$ in $G(\{1, a, a+1\}^2)$.

2. Vertices 0, 1, 2, 3, 4 are colored with the colors 1, 2, 3, 4 and 1, respectively.

Again considering cliques $C(2)$ and $C(3)$ in $G(\{1, a, a+1\}^2)$, we get

$$\{\lambda(1-a), \lambda(2+a)\} \subseteq \{5, 6, 7\},$$

and

$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{5, 6, 7\},$$

a contradiction since vertices $1-a$, $2-a$, $a+2$ and $a+3$ induce a clique in $G(\{1, a, a+1\}^2)$.

3. Vertices 0, 1, 2, 3, 4, 5 are colored with the colors 1, 2, 3, 4, 5 and 1, respectively. Considering the cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1, a, a+1\}^2)$, we get

$$\begin{aligned}\{\lambda(-a), \lambda(1-a), \lambda(1+a), \lambda(2+a)\} &= \{4, 5, 6, 7\}, \\ \{\lambda(2-a), \lambda(3+a)\} &\subseteq \{1, \lambda(-a), \lambda(1+a)\} \setminus \{4, 5\}, \\ \{\lambda(3-a), \lambda(4+a)\} &\subseteq \{2, \lambda(1-a), \lambda(2+a)\} \setminus \{4, 5\},\end{aligned}$$

and thus

$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{1, 6, 7\} \quad \text{and} \quad \{\lambda(3-a), \lambda(4+a)\} \subseteq \{2, 6, 7\}.$$

Assuming that none of cases 1 or 2 occurs, we have two subcases to consider:

- (a) $\lambda(6) = 2$.

Considering the clique $C(4)$ in $G(\{1, a, a+1\}^2)$, we get

$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{3, \lambda(2-a), \lambda(3+a)\} \setminus \{1, 2\} = \{3, 6, 7\}.$$

If $\{\lambda(4-a), \lambda(5+a)\} = \{3, 6\}$, then

$$\begin{aligned}\{\lambda(3-a), \lambda(4+a)\} &= \{2, 7\}, \\ \{\lambda(2-a), \lambda(3+a)\} &= \{1, 6\}, \\ \{\lambda(1-a), \lambda(2+a)\} &= \{5, 7\}\end{aligned}$$

and

$$\{\lambda(-a), \lambda(1+a)\} = \{4, 6\}.$$

If $\lambda(-a) = 6$ then $\lambda(2-a) = 1$ and thus $\lambda(4-a) = \lambda(-a) = 6$ which corresponds to subcase 2. If $\lambda(1+a) = 6$ then $\lambda(3+a) = 1$ and thus $\lambda(5+a) = \lambda(1+a) = 6$ which again corresponds to subcase 2.

The case $\{\lambda(4-a), \lambda(5+a)\} = \{3, 7\}$ is similar and leads to the same conclusion.

Finally, if $\{\lambda(4-a), \lambda(5+a)\} = \{6, 7\}$ then $\lambda(3-a) = \lambda(4+a) = 1$, a contradiction since $d_{G(\{1, a, a+1\})}(4-a, 5+a) = 2$.

- (b) $\lambda(6) = 6$.

Considering the clique $C(4)$ in $G(\{1, a, a+1\}^2)$, we get

$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{3, \lambda(2-a), \lambda(3+a)\} \setminus \{1, 6\} = \{3, 7\}.$$

This implies

$$\begin{aligned}\{\lambda(3-a), \lambda(4+a)\} &= \{2, 6\}, \\ \{\lambda(2-a), \lambda(3+a)\} &= \{1, 7\}, \\ \{\lambda(1-a), \lambda(2+a)\} &= \{5, 6\}\end{aligned}$$

and

$$\{\lambda(-a), \lambda(1+a)\} = \{4, 7\}.$$

If $\lambda(-a) = 7$ then $\lambda(2-a) = 1$ and thus $\lambda(4-a) = \lambda(-a) = 7$ which corresponds to subcase 2. If $\lambda(1+a) = 7$ then $\lambda(3+a) = 1$ and thus $\lambda(5+a) = \lambda(1+a) = 7$ which again corresponds to subcase 2.

4. Vertices $0, 1, 2, 3, 4, 5, 6$ are colored with the colors $1, 2, 3, 4, 5, 6$ and 1 , respectively. Again considering the cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1, a, a+1\}^2)$, we get

$$\begin{aligned}\{\lambda(-a), \lambda(1-a), \lambda(1+a), \lambda(2+a)\} &= \{4, 5, 6, 7\}, \\ \{\lambda(2-a), \lambda(3+a)\} &\subseteq \{1, \lambda(-a), \lambda(1+a)\} \setminus \{4, 5\},\end{aligned}$$

and thus

$$\{\lambda(3-a), \lambda(4+a)\} \subseteq \{2, \lambda(1-a), \lambda(2+a)\} \setminus \{4, 5, 6\} = \{2, 7\}.$$

This implies

$$\begin{aligned}\{\lambda(2-a), \lambda(3+a)\} &= \{1, 6\}, \\ \{\lambda(1-a), \lambda(2+a)\} &= \{5, 7\}\end{aligned}$$

and

$$\{\lambda(-a), \lambda(1+a)\} = \{4, 6\}.$$

Therefore,

$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{3, \lambda(2-a), \lambda(3+a)\} \setminus \{1, 6\} = \{3\},$$

a contradiction since $d_{G(\{1, a, a+1\})}(4-a, 5+a) = 2$.

Therefore, every 2-distance 7-coloring λ of $G(\{1, a, a+1\})$ is necessarily 7-periodic, and thus $\chi_2(G(\{1, a, a+1\})) = 7$ if and only if 7 do not divide any element of $\{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$. This is clearly the case if and only if $a \equiv 2 \pmod{7}$ or $a \equiv 4 \pmod{7}$. \square

The following result provides an upper bound on $\chi_2(G(\{1, a, a+1\}))$ for any value of a .

Theorem 8 *For every integer a , $a \geq 3$, $\chi_2(G(\{1, a, a+1\})) \leq 9 = \Delta(G(\{1, a, a+1\})) + 3$.*

Proof. We consider three cases, according to the value of $(a \pmod{3})$:

1. $a = 3k$, $k \geq 1$.

Let λ be the $3a$ -periodic mapping defined by the pattern

$$(123)^k(456)^k(789)^k.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 9$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k-1\} \cup \{3ap + 3q, p \geq 1, 1-k \leq q \leq k-1\}.$$

Therefore, $d(x, y) \notin \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$, and thus λ is a 2-distance 9-coloring of $G(\{1, a, a+1\})$.

2. $a = 3k+1$, $k \geq 1$.

Let λ be the $(3a+2)$ -periodic mapping defined by the pattern

$$(123)^k(456)^k7123(789)^{k-1}4568.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 6$, then

$$\begin{aligned}d(x, y) \in & \{3q, 0 \leq q \leq k-1\} \\ & \cup \{3q + 2a - 1, 1-k \leq q \leq 0\} \\ & \cup \{(3a+2)p + 2a - 1, p > 0\} \\ & \cup \{(3a+2)p - 2a + 1, p > 0\} \\ & \cup \{(3a+2)p + 3q, p > 0, 1-k \leq q < 0\} \\ & \cup \{(3a+2)p + 3q + 2a - 1, p > 0, 1-k \leq q < 0\} \\ & \cup \{(3a+2)p + 3q, p > 0, 0 < q \leq k-1\} \\ & \cup \{(3a+2)p + 3q - 2a + 1, p > 0, 0 < q \leq k-1\}.\end{aligned}$$

If $\lambda(x) = \lambda(y) = 7$, then

$$\begin{aligned} d(x, y) \in & \{3q, 0 \leq q \leq k-2\} \\ & \cup \{3q+4, 0 \leq q \leq k-2\} \\ & \cup \{(3a+2)p+3q-4, p > 0, 2-k \leq q \leq 0\} \\ & \cup \{(3a+2)p+3q+4, p > 0, 0 \leq q \leq k-2\} \\ & \cup \{(3a+2)p+3q, p > 0, 2-k \leq q \leq k-2\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = 8$, then

$$\begin{aligned} d(x, y) \in & \{3q, 0 \leq q \leq k-2\} \\ & \cup \{3q+a-2, 2-k \leq q \leq 0\} \\ & \cup \{(3a+2)p+a-2, p > 0\} \\ & \cup \{(3a+2)p-a+2, p > 0\} \\ & \cup \{(3a+2)p+3q, p > 0, 2-k \leq q < 0\} \\ & \cup \{(3a+2)p+3q+a-2, p > 0, 2-k \leq q < 0\} \\ & \cup \{(3a+2)p+3q, p > 0, 0 < q \leq k-2\} \\ & \cup \{(3a+2)p+3q-a+2, p > 0, 0 < q \leq k-2\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = 9$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k-2\} \cup \{(3a+2)p+3q, p \geq 1, 2-k \leq q \leq k-2\}.$$

Therefore, in all these cases, $d(x, y) \notin \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$, and thus λ is a 2-distance 9-coloring of $G(\{1, a, a+1\})$.

3. $a = 3k+2, k \geq 1$.

Let λ be the $(3a+1)$ -periodic mapping defined by the pattern

$$(123)^{k+1}(456)^{k+1}(789)^k 7.$$

If $\lambda(x) = \lambda(y) = c, 1 \leq c \leq 7$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k\} \cup \{(3a+1)p+3q, p \geq 1, -k \leq q \leq k\}.$$

If $\lambda(x) = \lambda(y) = c, 8 \leq c \leq 9$, then

$$d(x, y) \in \{3q, 0 \leq q \leq k-1\} \cup \{(3a+1)p+3q, p \geq 1, 1-k \leq q \leq k-1\}.$$

Therefore, in both cases, $d(x, y) \notin \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$, and thus λ is a 2-distance 9-coloring of $G(\{1, a, a+1\})$.

This concludes the proof. □

From Theorems 7 and 8, we thus get:

Corollary 9 *For every integer $a, a \geq 3, a \not\equiv 2, 4 \pmod{7}$,*

$$8 \leq \chi_2(G(\{1, a, a+1\})) \leq 9.$$

5 The case $D = \{1, \dots, m, a\}, 2 \leq m < a$

We study in this section the 2-distance chromatic number of distance graphs $G(D)$ with $D = \{1, \dots, m, a\}, 2 \leq m < a$ (note that the case $a = m+1$ is already solved by Proposition 5).

When $D = \{1, \dots, m, a\}$, we have $\Delta(G(D)) = 2m+2$ and

$$D^2 = \{1, 2, \dots, 2m\} \cup \{a-m, a-m+1, \dots, a+m\} \cup \{2a\}.$$

We first prove the following:

Theorem 10 For all integers m and a , $2 \leq m < a$,

$$\chi_2(G(\{1, \dots, m, a\})) = 2m + 3 = \Delta(G(\{1, \dots, m, a\})) + 1$$

if and only if $a \equiv m + 1 \pmod{2m + 3}$ or $a \equiv m + 2 \pmod{2m + 3}$.

Proof. Since $\{1, \dots, m, a\}^2 = \{1, \dots, 2m\} \cup \{a - m, a - m + 1, \dots, a + m\} \cup \{2a\}$, $d \not\equiv 0 \pmod{2m + 3}$ for every $d \in \{1, \dots, m, a\}^2$ whenever $a \equiv m + 1 \pmod{2m + 3}$ or $a \equiv m + 2 \pmod{2m + 3}$, and thus, by Proposition 1 and Observation 4, $\chi_2(G(\{1, \dots, m, a\})) = 2m + 3$.

We now claim that every 2-distance $(2m + 3)$ -coloring λ of $G(\{1, \dots, m, a\})$ is necessarily $(2m + 3)$ -periodic, that is $\lambda(x + 2m + 3) = \lambda(x)$ for every $x \in \mathbb{Z}$. To show that, it suffices to prove that any $2m + 3$ consecutive vertices $x, \dots, x + 2m + 2$ must be assigned distinct colors. Assume to the contrary that this is not the case and, without loss of generality, let $x = 0$. Since vertices $0, 1, \dots, 2m$ necessarily get distinct colors, we only have two cases to consider:

1. Vertices $0, 1, \dots, 2m + 1$ are colored with the colors $1, 2, \dots, 2m + 1$ and 1 , respectively.
Note that vertices $m - a$ and $m + a$ are both adjacent to all vertices $0, 1, \dots, 2m$. Hence,

$$\{\lambda(m - a), \lambda(m + a)\} = \{2m + 2, 2m + 3\},$$

which implies

$$\{\lambda(m + 1 - a), \lambda(m + 1 + a)\} = \{2m + 2, 2m + 3\}$$

(more precisely, $\lambda(m + 1 - a) = 4m + 5 - \lambda(m - a)$ and $\lambda(m + 1 + a) = 4m + 5 - \lambda(m + a)$). This implies $\lambda(m + 2 - a) = \lambda(m + 2 + a) = 2$, a contradiction since $d(m + 2 - a, m + 2 + a) = 2$.

2. Vertices $0, 1, \dots, 2m + 1, 2m + 2$ are colored with the colors $1, 2, \dots, 2m + 1, 2m + 2$ and 1 , respectively.

As in the previous case we have

$$\{\lambda(m - a), \lambda(m + a)\} = \{2m + 2, 2m + 3\},$$

which implies

$$\{\lambda(m + 1 - a), \lambda(m + 1 + a)\} = \{1, 2m + 3\}.$$

We thus get $\lambda(m + 2 - a) = \lambda(m + 2 + a) = 2$, again a contradiction.

Therefore, every 2-distance $(2m + 3)$ -coloring λ of $G(\{1, \dots, m, a\})$ is necessarily $(2m + 3)$ -periodic, and thus $\chi_2(G(\{1, \dots, m, a\})) = 2m + 3$ if and only if $2m + 3$ do not divide any element of $\{1, 2, \dots, 2m\} \cup \{a - m, a - m + 1, \dots, a + m\} \cup \{2a\}$. This is clearly the case if and only if $a \equiv m + 1 \pmod{2m + 3}$ or $a \equiv m + 2 \pmod{2m + 3}$. □

For other values of a , we propose the following general upper bound on

Theorem 11 For all integers m and a , $2 \leq m < a$,

$$\chi_2(G(\{1, \dots, m, a\})) \leq 4m + 2 = 2\Delta(G(\{1, \dots, m, a\})) - 2.$$

Proof. Let $a = (2m + 1)k + r$, $0 \leq r < 2m + 1$. We consider four cases, depending on the value of r . In each case, we will provide a periodic 2-distance $(4m + 2)$ -coloring of the distance graph $G(\{1, \dots, m, a\})$.

1. $r < m$.

Let λ be the $(2a - r - m)$ -periodic mapping defined by the pattern

$$[12 \dots (2m + 1)]^k [(2m + 2)(2m + 1) \dots (4m + 2)]^{k-1} (2m + 2)(2m + 3) \dots (3m + r + 2).$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 3m + r + 2$, then

$$d(x, y) \in \begin{aligned} &\{q(2m + 1), 0 \leq q \leq k - 1\} \\ &\cup \{p(2a - r - m) + q(2m + 1), p \geq 1, 1 - k \leq q \leq k - 1\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = c$, $3m + r + 3 \leq c \leq 4m + 2$, then

$$d(x, y) \in \begin{aligned} &\{q(2m + 1), 0 \leq q \leq k - 2\} \\ &\cup \{p(2a - r - m) + q(2m + 1), p \geq 1, 2 - k \leq q \leq k - 2\}. \end{aligned}$$

Therefore, in both cases, $d(x, y) \notin \{1, 2, \dots, 2m\} \cup \{a - m, a - m + 1, \dots, a + m\} \cup \{2a\}$, and thus λ is a 2-distance $(4m + 2)$ -coloring of $G(\{1, \dots, m, a\})$.

2. $r = m$.

Let λ be the $(2a - 2m)$ -periodic mapping defined by the pattern

$$[12 \dots (2m + 1)]^k [(2m + 2)(2m + 1) \dots (4m + 2)]^k.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 4m + 2$, then

$$d(x, y) \in \begin{aligned} &\{q(2m + 1), 0 \leq q \leq k - 1\} \\ &\cup \{p(2a - 2m) + q(2m + 1), p \geq 1, 1 - k \leq q \leq k - 1\}. \end{aligned}$$

Therefore, $d(x, y) \notin \{1, 2, \dots, 2m\} \cup \{a - m, a - m + 1, \dots, a + m\} \cup \{2a\}$, and thus λ is a 2-distance $(4m + 2)$ -coloring of $G(\{1, \dots, m, a\})$.

3. $r = m + 1$.

Let λ be the $(2a + 1)$ -periodic mapping defined by the pattern

$$[12 \dots (2m + 1)]^{k+1} [(2m + 2)(2m + 1) \dots (4m + 2)]^k (2m + 2)(2m + 3).$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 2m + 3$, then

$$d(x, y) \in \begin{aligned} &\{q(2m + 1), 0 \leq q \leq k\} \\ &\cup \{p(2a + 1) + q(2m + 1), p \geq 1, -k \leq q \leq k\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = c$, $2m + 4 \leq c \leq 4m + 2$, then

$$d(x, y) \in \begin{aligned} &\{q(2m + 1), 0 \leq q \leq k - 1\} \\ &\cup \{p(2a + 1) + q(2m + 1), p \geq 1, 1 - k \leq q \leq k - 1\}. \end{aligned}$$

Therefore, in both cases, $d(x, y) \notin \{1, 2, \dots, 2m\} \cup \{a - m, a - m + 1, \dots, a + m\} \cup \{2a\}$, and thus λ is a 2-distance $(4m + 2)$ -coloring of $G(\{1, \dots, m, a\})$.

4. $m + 2 \leq r < 2m + 1$.

Let λ be the $(2a - r + m + 1)$ -periodic mapping defined by the pattern

$$[12 \dots (2m + 1)]^{k+1} [(2m + 2)(2m + 1) \dots (4m + 2)]^k (2m + 2)(2m + 3) \dots (m + r + 1).$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq m + r + 1$, then

$$d(x, y) \in \begin{aligned} &\{q(2m + 1), 0 \leq q \leq k\} \\ &\cup \{p(2a - r + m + 1) + q(2m + 1), p \geq 1, -k \leq q \leq k\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = c$, $m + r + 2 \leq c \leq 4m + 2$, then

$$d(x, y) \in \begin{aligned} &\{q(2m + 1), 0 \leq q \leq k - 1\} \\ &\cup \{p(2a - r + m + 1) + q(2m + 1), p \geq 1, 1 - k \leq q \leq k - 1\}. \end{aligned}$$

Therefore, in both cases, $d(x, y) \notin \{1, 2, \dots, 2m\} \cup \{a - m, a - m + 1, \dots, a + m\} \cup \{2a\}$, and thus λ is a 2-distance $(4m + 2)$ -coloring of $G(\{1, \dots, m, a\})$.

This concludes the proof. □

From Theorems 10 and 11, we thus get:

Corollary 12 *For all integers m and a , $2 \leq m < a$, $a \not\equiv m+1, m+2 \pmod{2m+3}$,*

$$2m+4 \leq \chi_2(G(\{1, \dots, m, a\})) \leq 4m+2.$$

6 Discussion

In this paper, we studied 2-distance colorings of several types of distance graphs. In each case, we characterized those distance graphs that admit an optimal 2-distance coloring, that is distance graphs $G(D)$ with $\chi_2(G(D)) = \Delta(G(D)) + 1$. We also provided general upper bounds for the 2-distance chromatic number of the considered graphs.

We leave as open problems the question of completely determining the 2-distance chromatic number of distance graphs $G(D)$ when $D = \{1, a, a+1\}$, $a \geq 3$, or $D = \{1, \dots, m, a\}$, $2 \leq m < a$.

Considering other types of sets D would certainly be also an interesting direction for future research.

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References

- [1] G. Agnarsson and M.M. Halldórsson. Coloring powers of planar graphs. *SIAM J. Discrete Math.* 16(4):651–662 (2003).
- [2] M. Bonamy, B. Lévêque and A. Pinlou. 2-distance coloring of sparse graphs. *J. Graph Theory* 77(3):190–218 (2014).
- [3] M. Bonamy, B. Lévêque and A. Pinlou. Graphs with maximum degree $\Delta \geq 17$ and maximum average degree less than 3 are list 2-distance $(\Delta+2)$ -colorable. *Discrete Math.* 317:19–32 (2014).
- [4] O.V. Borodin and A.O. Ivanova. 2-distance $(\Delta+2)$ -coloring of planar graphs with girth six and $\Delta \geq 18$. *Discrete Math.* 309:6496–6502 (2009).
- [5] T. Calamoneri. The $L(h, k)$ -labelling problem: An updated survey and annotated bibliography. *The Computer Journal* 54(8):1344–1371 (2011).
- [6] Z. Dvořák, D. Král, P. Nejedlý, and R. Škrekovski. Coloring squares of planar graphs with girth six. *European J. Combin.* 29(4):838–849 (2008).
- [7] F. Kramer and H. Kramer. Un problème de coloration des sommets d'un graphe. *C. R. Acad. Sci. Paris A* 268(7):46–48 (1969).
- [8] F. Kramer and H. Kramer. A survey on the distance-coloring of graphs. *Discrete Math.* 308:422–426 (2008).
- [9] K.W. Lih and W.F. Wang. Coloring the square of an outerplanar graph. *Taiwan. J. Math.* 10(4):1015–1023 (2006).
- [10] D. D.-F. Liu. From rainbow to the lovely runner: A survey on coloring parameters of distance graphs. *Taiwanese J. Math.* 12(4):851–871 (2008).

- [11] H. Walther. Über eine spezielle Klasse unendlicher Graphen. In K. Wagner and R. Bodendiek, eds, *Graphentheorie*, vol. 2, pp. 268–295 (1990), Bibl. Inst., Mannheim.
- [12] W.F. Wang and K.W. Lih. Labeling planar graphs with conditions on girth and distance two. *SIAM J. Discrete Math.* 17(2):264–275 (2003).
- [13] G. Wegner. Graphs with given diameter and a colouring problem. Technical Report, University of Dortmund (1977).
- [14] X. Zhu. Circular chromatic number of distance graphs with distance sets of cardinality three. *J. Graph Theory* 41(3):195–207 (2002).